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A PRESENTATION FOR K_2 OF SPLIT RADICAL PAIRS

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Introduction

By a split radical pair we mean a ring R (always commutative with 1) together with an ideal I contained in the Jacobson radical of R, such that the projection $R \to R/I$ admits a section. One of the main results of this paper is a description of the kernel of the map $K_2(R) \to K_2(R/I)$ for a split radical pair (R, I). This description is based on elements $\langle a, b\rangle$ in $K_2(R)$, defined for $a, b \in R$ with $1 + ab \in R^*$. These symbols were introduced by Dennis and Stein [8] who also gave a list of relations. We use some of their relations, written in a manageable form, to define groups $D(R)$ and $D(R, I)$ (see Section 2). It turns out that for a split radical pair the sequence

$$1 \to D(R, I) \to K_2(n, R) \to K_2(n, R/I) \to 1$$

is exact for $n \geq 3$. In some cases it is possible to compute $D(R, I)$ in terms of differential forms which gives new proofs for results of Van der Kallen on dual numbers [5] and of Bloch on Artinian $\mathbb{Q}$-algebras [1]; see (3.12). For some types of rings one can show the equality of $K_2(R)$ and $D(R)$. For discrete valuation rings the presentation of Dennis and Stein in [3] can be reformulated that way (see (5.1) and (5.2)). Then with the help of a reduction theorem (4.2) we can transfer this result to some other types of rings.

All these results were announced in [6].

Using a different method Van der Kallen is able to show the equality of $D(R)$ and $K_2(R)$ for many other rings. Those results were also in the announcement [6]. Recently Van der Kallen obtained new proofs. These will be published elsewhere.

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1. Elements of K_2 and the Steinberg group

1.1. Definitions (cf. [7, §5]). Let R be a ring and n an integer ≥ 3. The Steinberg group of rank n of R, denoted by $\mathrm{St}(n, R)$, is defined as the group with generators $x_{ij}(a)$ where $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$ and $a \in R$, subject to the relations

(R1) $\quad x_{ij}(a)x_{ij}(b) = x_{ij}(a+b)$

(R2) $$[x_{ij}(a), x_{kl}(b)] = \begin{cases} 1 & \text{if } k \neq j \text{ and } l \neq i \\ x_{il}(ab) & \text{if } k = j \text{ and } l \neq i \end{cases}$$

(in a group $[x, y]$ denotes the commutator $xyx^{-1}y^{-1}$).

The group $E(n, R)$ is the subgroup of $\mathrm{Gl}(n, R)$ generated by the elementary matrices $e_{ij}(a)$ with $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$ and $a \in R$. The matrix $e_{ij}(a)$ has 1's on the diagonal, a on the (i, j)-spot and zeroes elsewhere.

There is a homomorphism $\phi : \mathrm{St}(n, R) \to E(n, R)$ sending $x_{ij}(a)$ to $e_{ij}(a)$. The kernel of ϕ is denoted by $K_2(n, R)$.

If $m > n$, there are homomorphisms $\mathrm{St}(n, R) \to \mathrm{St}(m, R)$ and $E(n, R) \to E(m, R)$ sending $x_{ij}(a)$ and $e_{ij}(a)$ to the elements with the same names. These homomorphisms commute with the ϕ's, so that there is a homomorphism $K_2(n, R) \to K_2(m, R)$. Taking the limits one gets the groups $\mathrm{St}(R)$, $E(R)$ and $K_2(R)$ and an exact sequence

$$1 \to K_2(R) \to \mathrm{St}(R) \to E(R) \to 1.$$

A ringhomomorphism $f : R \to S$ gives rise to homomorphisms $\mathrm{St}(n, R) \to \mathrm{St}(n, S)$ and $E(n, R) \to E(n, S)$ sending $x_{ij}(a)$ and $e_{ij}(a)$ to $x_{ij}(f(a))$ and $e_{ij}(f(a))$ respectively. These homomorphisms commute with the ϕ's and hence induce a homomorphism $K_2(n, R) \to K_2(n, S)$. With this definition $\mathrm{St}(\ )$, $E(\ )$ and $K_2(\ )$ become functors from the category of rings to the category of groups and all homomorphisms mentioned before become natural transformations.

Let us now consider the category which has as its objects pairs (R, I) consisting of a ring R and a (not necessarily proper) ideal I of R, and in which a morphism $f : (R, I) \to (S, J)$ is a ringmorphism $f : R \to S$ such that $f(I) \subset J$. The category of rings is embedded in this new category via $R \to (R, R)$. If one has a group valued functor F on the category of rings that maps the trivial ring to the trivial group, one can extend it to the category of pairs by $F(R, I) = \ker(F(R) \to F(R/I))$. In this way we get the groups $\mathrm{St}(n, R, I)$, $\mathrm{St}(R, I)$, $E(n, R, I)$, $E(R, I)$, $K_2(n, R, I)$ and $K_2(R, I)$. We emphasize that our $K_2(R, I)$ is not the $K_2(I)$ of Milnor [7, §6]; we will come back to this difference in (3.11).

1.2. Fix a ring R and an integer $n \geq 3$. For $u \in R^*$ one defines $w_{ij}(u) = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)$ and $h_{ij}(u) = w_{ij}(u)w_{ij}(-1)$ in $\mathrm{St}(n, R)$. The element $h_{ij}(u)$ is mapped by ϕ to the diagonal matrix with u on the (i, i)-spot, u^{-1} on the (j, j)-spot and 1 elsewhere on the diagonal.

An important role in the theory of K_2 is played by the following elements of $K_2(n, R)$:

$$\{u, v\}_{ij} = h_{ij}(uv)h_{ij}(u)^{-1}h_{ij}(v)^{-1} \quad \text{for } u, v \in R^*$$

and

$$\langle a, b\rangle_{ij} = x_{ji}(-b(1+ab)^{-1})x_{ij}(a)x_{ji}(b)x_{ij}(-a(1+ab)^{-1})h_{ij}^{-1}(1+ab)$$

for $a, b \in R$ such that $1 + ab \in R^*$.

Both types of elements are central in $\mathrm{St}(n, R)$ and independent of the index (see [4, §9]). One could therefore omit the index. We prefer to use a rudimentary index $*$ in order to distinguish between the elements introduced here and some unadorned $\{u, v\}$ and $\langle a, b\rangle$ that are considered in Section 2.

From the identities with these elements listed in [4, §9] we want to mention at this moment only the following three:

$$\langle a, b\rangle_* = \langle -b, -a\rangle_*^{-1}$$

$$\langle a, b\rangle_* \langle a, c\rangle_* = \langle a, b + c + abc\rangle_*$$

$$\langle a, bc\rangle_* = \langle ab, c\rangle_* \langle ac, b\rangle_*.$$

The second one is not in [4] in this form but it is easily obtained from what is called there (H2):

$$\begin{aligned}
&\langle a, b + c + abc\rangle_* \\
&\quad = \langle a, b\rangle_* \langle a(1+ab)^{-1}, c(1+ab)\rangle_* \{1+ab, (1+ab+ac)(1+ab)^{-1}\}_* \\
&\quad = \langle a, b\rangle_* \langle a, c\rangle_* \langle ac(1+ab)^{-1}, 1+ab\rangle_* \langle -(1+ab), -ac(1+ab)^{-1}\rangle_* \\
&\quad = \langle a, b\rangle_* \langle a, c\rangle_*.
\end{aligned}$$

1.3. If one writes relation (R2) as $x_{ij}(a)x_{jl}(b) = x_{jl}(b)x_{ij}(a)x_{il}(ab)$, then three x's appear at the right-hand side as opposed to two at the left-hand side. This turned out to be unpleasant for our reconstruction of (part of) the Steinberg group. However the observation that $x_{ij}(a)$ and $x_{il}(ab)$ have the same first index made us consider the following elements: For $i \in \{1, 2, \ldots, n\}$ and $\boldsymbol{a} = (a_1, \ldots, a_n)$ a sequence in R with $a_i = 0$ we put:

$$x_i(\boldsymbol{a}) = x_{i1}(a_1)x_{i2}(a_2)\cdots x_{i\,i-1}(a_{i-1})x_{i\,i+1}(a_{i+1})\cdots x_{in}(a_n).$$

Furthermore:

For $\boldsymbol{u} = (u_1, \ldots, u_{n-1})$ a sequence in R^* we put

$$h(\boldsymbol{u}) = h_{12}(u_1)h_{23}(u_2)\cdots h_{n-1\,n}(u_{n-1}).$$

It is readily seen that $\phi : \mathrm{St}(n, R) \to E(n, R)$ maps $x_i(a)$ to the matrix

$$e_i(a) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ a_1 & \cdots & a_{i-1} & 1 & a_{i+1} & \cdots & a_n \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \quad \text{(off the diagonal and the } i\text{th row are zeroes)}$$

and $h(u)$ to the matrix

$$\tilde{h}(u) = \mathrm{diag}(u_1, u_1^{-1}u_2, u_2^{-1}u_3, \ldots, u_{n-2}^{-1}u_{n-1}, u_{n-1}^{-1}).$$

Notation. $\Omega_n = \{(a_{ij}) \in E(n, R) \mid \det(a_{ij})_{\substack{1 \le i \le k \\ 1 \le j \le k}} \in R^* \text{ for } k = 1, \ldots, n\}$

$$\tilde{\Omega}_n = \phi^{-1}\Omega_n.$$

1.4. Proposition. *Any matrix in Ω_n can be written uniquely as a product*

$$e_n(a^{(n)})e_{n-1}(a^{(n-1)}) \cdots e_1(a^{(1)})\tilde{h}(u)$$

where a factor $e_k(a^{(k)})$ only occurs if $a^{(k)} \neq (0, \ldots, 0)$ and $\tilde{h}(u)$ only occurs if $u \neq (1, \ldots, 1)$. The empty product, of course, represents the identity matrix. Conversely, any such product represents an element of Ω_n.

Proof. The matrix $e_n(a^{(n)}) \cdots e_k(a^{(k)})$ has off the diagonal zeroes in the first $(k-1)$ rows and $a^{(k)}$ in the kth row. The diagonal entries in these rows are 1. Furthermore the determinant of the upper left-hand $i \times i$-matrix in the product $e_n(a^{(n)}) \cdots e_1(a^{(1)})\tilde{h}(u)$ is u_i. These observations prove the uniqueness and the fact that such a product is in Ω_n; moreover they indicate how to construct the desired product decomposition for a given matrix from Ω_n. □

1.5. Corollary. *Any element of $\tilde{\Omega}_n$ can be written uniquely as a product*

$$x_n(a^{(n)})x_{n-1}(a^{(n-1)}) \cdots x_1(a^{(1)})h(u)d \quad \text{with } d \in K_2(n, R).$$

Conversely, such a product is in $\tilde{\Omega}_n$. □

1.6. We now consider the computational aspects of the elements $x_i(a)$ and $h(u)$. We write $\mathbf{0} = (0, \ldots, 0)$, $\mathbf{1} = (1, \ldots, 1)$ and define addition and multiplication of sequences in R coordinatewise.

Proposition. (i) *$x_i(\mathbf{0}) = 1$ for all i.*

(ii) $h(\mathbf{1}) = 1$.

(iii) *$x_i(a)x_i(b) = x_i(a + b)$ for all i, a and b.*

(iv) $h(u)h(v) = h(uv)\{u_1, v_1\}_*\{v_1, u_2\}_*\{u_2, v_2\}_* \cdots \{v_{n-2}, u_{n-1}\}_*\{u_{n-1}, v_{n-1}\}_*$ *for all u, v.*

(v) $h(u)x_i(a) = x_i(b)h(u)$ *for all* i, a *and* u, *where* $b_k = a_k u_{k-1} u_k^{-1} u_{i-1}^{-1} u_i$ *for* $k = 1, \ldots, n$ (*put* $u_0 = u_n = 1$).

(vi) $x_i(a)x_j(b) = x_j(c)x_i(d)h(u)\langle(-1)^{i+j+1}a_j, (-1)^{i+j+1}b_i\rangle_*$ *for all* $i \neq j$, a *and* b *such that* $1 + a_j b_i \in R^*$, *where we write*

$$\begin{aligned}
c_k &= (b_k - a_k b_i)(1 + a_j b_i)^{-1} && \text{for } k \neq j,\\
d_k &= a_k + a_j b_k && \text{for } k \neq i, j,\\
d_j &= a_j(1 + a_j b_i), &&\\
u_k &= 1 + a_j b_i && \text{for } k = i, i+1, \ldots, j-1 \text{ if } i < j\\
& && \text{for } k = j, j+1, \ldots, i-1 \text{ if } i > j,\\
u_k &= 1 && \text{else.}
\end{aligned}$$

Proof. (i), (ii), (iii) are immediate from (R1) and (R2). (iv) follows from [7] lemmas (9.7) and (9.10). (v) is obvious from corollary (9.4) in [7].

(vi) Write $x_i(a) = x_i(a')x_{ij}(a_j)$ with $a'_j = 0$, and $x_j(b) = x_{ji}(b_i)x_j(b')$ with $b'_i = 0$. Then one has

$$x_i(a)x_j(b) = x_i(a')x_{ji}(b_i(1 + a_j b_i)^{-1})\, x_{ij}(a_j(1 + a_j b_i))h_{ij}(1 + a_j b_i)x_j(b')\langle a_j, b_i\rangle_*.$$

Now one can proceed using (R1), (R2) and (v). One sees that no further elements appear from $K_2(n, R)$. Next observe that $h_{ij}(1 + a_j b_i) = h(u)\{-1, 1 + a_j b_i\}_*^{i+j+1}$ (cf. [7] lemmas (9.10), (9.7), (9.8)) and that

$$\langle a_j, b_i\rangle_*\{(-1)^{i+j+1}, 1 + a_j b_i\}_* = \langle(-1)^{i+j+1}a_j, (-1)^{i+j+1}b_i\rangle_*$$

(cf. [4] §9 (b)–(c)).

This shows that the factor from $K_2(n, R)$ in (vi) is correct. The rest can be checked at the matrix level because of (1.4) and (1.5). □

2. The groups $D(R, I)$

2.1. Definition. "Let (R, I) be a pair" means that we want to consider a ring R together with an ideal $I \subset R$. A pair (R, I) is called split if the projection $R \to R/I$ splits. A pair (R, I) is called radical if I is contained in the Jacobson radical of R.

2.2. Definition. Let (R, I) be a pair. The group $D(R, I)$ has generators $\langle a, b\rangle$, one for each couple $(a, b) \in R \times I \cup I \times R$ such that $1 + ab \in R^*$, subject to the relations

(D0) $D(R, I)$ is abelian.

(D1) $\langle a, b\rangle\langle -b, -a\rangle = 1$.

(D2) $\langle a, b\rangle\langle a, c\rangle = \langle a, b + c + abc\rangle$.

(D3) $\langle a, bc\rangle = \langle ab, c\rangle\langle ac, b\rangle$.

We write $D(R)$ instead of $D(R,R)$.

2.3. Remark. In (D3) it is possible that ab, ac, bc are in I while a, b and c are not. In [6] we required (D3) only for the case that one of the elements a, b, c is in I. For a split radical pair and for (R,R) this makes no difference as we shall see below, but in general there is a difference. Here is an example:

Take $R = \mathbb{F}_2[t]/(t^3)$, $I = (t^2)$. If one allows (D3) in the present form, it follows that $\langle t, t^2\rangle = \langle t^2, t\rangle\langle t^2, t\rangle = \langle t^2, 0\rangle = 1$ (for the last step see Lemma (2.5)). If, on the other hand, (D3) is only required in the case where a, b or c is divisible by t^2, one defines a homomorphism onto $\mathbb{F}_2^+$ by $\langle \alpha + \beta t + \gamma t^2, \delta t^2\rangle \to \beta\delta$ and $\langle \delta t^2, \alpha + \beta t + \gamma t^2\rangle \to \beta\delta$. This homomorphism sends $\langle t, t^2\rangle$ to 1, the non-trivial element of $\mathbb{F}_2^+$.

Now we prove that for a split radical pair there is no difference. Let $R = S \oplus I$, where S is a ring and I an ideal of R contained in the Jacobson radical. Let $a = x + i$, $b = y + j$, $c = z + k$ with $x, y, z \in S$ and $i, j, k \in I$.

Suppose $xy = yz = xz = 0$. The following computation does not use the stronger form of (D3):

$$\langle a, bc\rangle = \langle x, bc\rangle\langle i(1+xbc)^{-1}, bc\rangle \quad \text{by (D1) and (D2).}$$

$$\langle ab, c\rangle = \langle xb, c\rangle\langle ib(1+xbc)^{-1}, c\rangle$$

$$\langle ac, b\rangle = \langle xc, b\rangle\langle ic(1+xbc)^{-1}, b\rangle$$

$$\langle i(1+xbc)^{-1}, bc\rangle = \langle ib(1+xbc)^{-1}, c\rangle\langle ic(1+xbc)^{-1}, b\rangle \quad \text{by (D3) (weak form).}$$

Combining these identities we see that $\langle a, bc\rangle = \langle ab, c\rangle\langle ac, b\rangle$ is equivalent to $\langle x, bc\rangle = \langle xb, c\rangle\langle xc, b\rangle$, i.e. we may assume $i = 0$. Similarly one may assume $j = 0$ and $k = 0$. Then the problem reads $\langle x, yz\rangle = \langle xy, z\rangle\langle xz, y\rangle$ or rather $\langle x, 0\rangle = \langle 0, z\rangle\langle 0, y\rangle$. But $\langle x, 0\rangle$, $\langle 0, z\rangle$ and $\langle 0, y\rangle$ are all trivial by (D2).

The conclusion is that for a split radical pair the strong form of (D3) follows from the weaker one.

2.4. For every $n \geq 3$ and every pair (R, I) there is a homomorphism $\delta : D(R, I) \to K_2(n, R, I)$ given by $\delta\langle a, b\rangle = \langle a, b\rangle_*$, as one sees from (1.2).

$D(\ ,\ )$ is a functor from the category of pairs to the category of (abelian) groups and δ is a natural transformation.

2.5. Lemma. *Let (R, I) be a pair. In $D(R, I)$ one has:*

(i) $\langle a, e\rangle = 1$ *if e is idempotent, in particular* $\langle a, 0\rangle = 1$ *and* $\langle a, 1\rangle = 1$.

(ii) $\langle a, b\rangle^{-1} = \langle a, -b(1+ab)^{-1}\rangle$, $\langle a, b\rangle^{-1} = \langle -a(1+ab)^{-1}, b\rangle$.

Proof. (i) From $\langle ae, e\rangle = \langle ae, e\rangle\langle ae, e\rangle$ it follows $\langle ae, e\rangle = 1$. Use (D3) once more: $\langle a, e\rangle = \langle ae, e\rangle\langle ae, e\rangle = 1$.

(ii) $\langle a, b\rangle\langle a, -b(1+ab)^{-1}\rangle = \langle a, 0\rangle = 1$. And similarly for the other identity. □

2.6. Notation. $(1+I)^* = (1+I) \cap R^*$, the group of units of the form $1+i$ with $i \in I$.

2.7. Lemma. *For $u, v \in (1+I)^*$ the following identity holds in $D(R, I)$:*

$$\langle (u-1)v^{-1}, v \rangle = \langle -u, -(v-1)u^{-1} \rangle.$$

Proof. $\langle (u-1)v^{-1}, v\rangle = \langle -(u-1)v^{-1}u^{-1}, v\rangle^{-1} = \langle -(u-1)v^{-1}u^{-1}, -uv\rangle$ by (2.5)(ii). Similarly $\langle -u, -(v-1)u^{-1}\rangle = \langle uv, (v-1)u^{-1}v^{-1}\rangle$. Hence:

$$\begin{aligned} &\langle (u-1)v^{-1}, v\rangle^{-1}\langle -u, -(v-1)u^{-1}\rangle \\ &= \langle uv, (u-1)u^{-1}v^{-1}\rangle\langle uv, (v-1)u^{-1}v^{-1}\rangle \\ &= \langle uv, (uv-1)u^{-1}v^{-1}\rangle \qquad \text{by (D2)} \\ &= \langle -1, (uv-1)u^{-1}v^{-1}\rangle^{-1} \qquad \text{by (2.5)} \\ &= 1. \end{aligned}$$

□

2.8. Definition. For $u, v \in R^*$ with u or $v \in (1+I)^*$ define:

$$\begin{aligned} \{u, v\} &= \langle (u-1)v^{-1}, v\rangle \qquad \text{if } u \in (1+I)^* \\ &= \langle -u, -(v-1)u^{-1}\rangle \qquad \text{if } v \in (1+I)^*. \end{aligned}$$

The previous lemma shows that there is no conflict if both u and v are in $(1+I)^*$.

2.9. Proposition. *The homomorphism $\delta : D(R, I) \to K_2(n, R, I)$ sends $\{u, v\}$ to $\{u, v\}_*$. In $D(R, I)$ the following identities are valid if the terms are defined:*

(i) $\{u, v\} = \{v, u\}^{-1}$
(ii) $\{u, vw\} = \{u, v\}\{u, w\}$
(iii) $\{u, -u\} = 1$
(iv) $\{u, 1-u\} = 1$.

Proof. The first assertion follows from [4, §9.c]. The identities are immediate from the definition in (2.8) and the relations in (2.2) and (2.5). □

Combining this proposition with Matsumoto's theorem [7, theorem 11.1] one sees

2.10. Corollary. *For a field F the homomorphism δ is bijective i.e.*

$$D(F) \simeq K_2(F) \simeq K_2(n, F) \quad \textit{for all } n \geq 3.$$

□

We conclude this section with some properties of $D(\ )$ as a functor.

2.11. Proposition. (i) *If $R = R_1 \times R_2$, a product of rings then*

$$D(R) \simeq D(R_1) \times D(R_2).$$

(ii) *If* $R = \varinjlim R_i$, *the limit of a filtered inductive system of rings, then*

$$D(R) = \varinjlim D(R_i).$$

(iii) *If* (R, I) *is a radical pair, then the sequence*

$$D(R, I) \to D(R) \to D(R/I) \to 0$$

is exact.

Proof. (i) Write an element of R as a couple (a_1, a_2) with $a_i \in R_i$ $(i = 1, 2)$. The homomorphism $D(R) \to D(R_1) \times D(R_2)$ is given by

$$\langle (a_1, a_2), (b_1, b_2) \rangle \mapsto (\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle).$$

It is easily checked that this is an isomorphism.

(ii) Clearly there is a homomorphism $\alpha : \varinjlim D(R_i) \to D(R)$. Consider a generator $\langle a, b \rangle$ of $D(R)$. For some index i the ring R_i contains a, b and $(1 + ab)^{-1}$. So there is an element $\langle a, b \rangle$ in $D(R_i)$ and it is mapped to $\langle a, b \rangle$ in $D(R)$. Similarly, each relation from the defining set of relations for $D(R)$ can be lifted to an identity in some $D(R_i)$. This shows that α is an isomorphism.

(iii) Let $a \to \bar{a}$ denote the projection $R \to R/I$. Since I is contained in the Jacobson radical of R, we have that $a \in R$ is a unit if and only if $\bar{a}$ is a unit in R/I. So for every $\langle \bar{a}, \bar{b} \rangle$ in $D(R/I)$ the element $\langle a, b \rangle$ is defined in $D(R)$. This $\langle a, b \rangle$ is mapped to $\langle \bar{a}, \bar{b} \rangle$. This shows exactness at $D(R/I)$. Let $D = \operatorname{im}(D(R, I) \to D(R))$. As the sequence $D(R, I) \to D(R) \to D(R/I)$ is a null sequence it is enough to prove that the map $\psi : D(R)/D \to D(R/I)$ has an inverse. First observe that for $s, t \in I$ and $1 + ab \in R^*$ one has

$$\begin{aligned} \langle a + s, b + t \rangle &= \langle a, b \rangle \langle s(1 + ab + at)^{-1}, b + t \rangle \langle a, t(1 + ab)^{-1} \rangle \\ &\equiv \langle a, b \rangle \bmod D. \end{aligned}$$

Hence the class of $\langle a, b \rangle$ modulo D depends only on the residue classes $\bar{a}$ and $\bar{b}$. Thus it makes sense to define

$$\zeta \langle \bar{a}, \bar{b} \rangle = \langle a, b \rangle \bmod D.$$

This map ζ clearly extends to a homomorphism $\zeta : D(R/I) \to D(R)/D$, which is inverse to ψ. □

Examples. (1). Let X be a scheme and let $D(O_X)$ be the sheaf (of abelian groups) associated to the presheaf $U \mapsto D(\Gamma(U, O_X))$. Then (2.11)(ii) states that for every point $x \in X$ there is an isomorphism: $D(O_X)_x \simeq D(O_{X,x})$.

(2). In (2.11)(iii) the map $D(R, I) \to D(R)$ may not be injective. Here is an example which appeared in a discussion with Spencer Bloch.

Let $R = \mathbb{F}_3[t]/(t^4)$ and $I = (t^3)$. There is a homomorphism $D(R, I) \to \mathbb{F}_3^+$ defined by $\langle \alpha + \beta t + \gamma t^2 + \delta t^3, \varepsilon t^3 \rangle \mapsto \beta\varepsilon$ and $\langle \varepsilon t^3, \alpha + \beta t + \gamma t^2 + \delta t^3 \rangle \mapsto -\beta\varepsilon$. It maps $\langle -t, t^3 \rangle$ to -1. In particular we see that $\langle -t, t^3 \rangle$ is a non-trivial element of $D(R, I)$. Now we compute $\langle -t, t^3 \rangle$ in $D(R)$:

$$\langle -t, t^3 \rangle = \langle -t^3, t \rangle \langle -t^2, t^2 \rangle = \langle -t^3, t \rangle^3.$$

Hence $\langle -t\ t^3 \rangle^4 = 1$. Moreover:

$$\langle -t, t^3 \rangle^3 = \langle -t, 3t^3 \rangle = 1.$$

So we see that in $D(R)$ the element $\langle -t, t^3 \rangle$ is trivial. This shows that the map $D(R, I) \to D(R)$ is not injective in this example.

3. The main theorem and its proof

3.1. Theorem. (i) *If (R, I) is a radical pair, then, for any integer $n \geq 3$, the homomorphism $\delta : D(R, I) \to K_2(n, R, I)$ is surjective. Consequently $\delta : D(R, I) \to K_2(R, I)$ is also surjective.*

(ii) *If (R, I) is a split radical pair, then for any integer $n \geq 3$, the homomorphism $\delta : D(R, I) \to K_2(n, R, I)$ is an isomorphism. Hence, $\delta : D(R, I) \to K_2(R, I)$ is an isomorphism too.*

The proof of this theorem fills the greater part of this section. In a game with words we construct a group $\mathcal{M}_\Delta$ and a homomorphism $\beta_\Delta : \mathcal{M}_\Delta \to \mathrm{St}(n, R, I)$. The map β_Δ turns out to be surjective. In the split case we exhibit an action of $\mathrm{St}(n, R/I)$ on $\mathcal{M}_\Delta$ and we show that the corresponding extension is isomorphic to $\mathrm{St}(n, R)$. From this it follows that β_Δ is an isomorphism. $D(R, I)$ will be equal to $\beta_\Delta^{-1}(K_2(n, R, I))$ and δ will be the restriction of β_Δ. Thus the results for β_Δ imply those for δ. The first part of the theorem is already known from [8, theorem 2.1]. Here we obtain a proof as a byproduct of the proof of the second part. From now on we fix a radical pair (R, I) and an integer $n \geq 3$.

3.2. Let π be a partial order relation on $\{1, 2, \ldots, n\}$ i.e. a subset of $\{(i, j) \mid 1 \leq i \leq n, i \leq j \leq n\}$ satisfying the following two conditions:

(1) both (i, j) and (j, i) are in π if and only if $i = j$.

(2) if $(i, j) \in \pi$ and $(j, k) \in \pi$ then $(i, k) \in \pi$.

By $\mathcal{W}_\pi$ we denote the set of words in the following letters

- $X_i(a)$ for $i \in \{1, 2, \ldots, n\}$ and $a = (a_1, \ldots, a_n)$ a sequence in R with $a_i = 0$ and $a_j \in I$ if $(i, j) \notin \pi$.
- $H(u)$ where $u = (u_1, \ldots, u_{n-1})$ is a sequence (of units) in $1 + I$.
- D where $D \in D(R, I)$.

Furthermore we define a map $\beta_\pi : \mathcal{W}_\pi \to \mathrm{St}(n, R)$ by (cf. (1.3) and (2.4)).

$$\beta_\pi(X_i(a)) = x_i(a), \quad \beta_\pi(H(u)) = h(u), \quad \beta_\pi(D) = \delta(D),$$

juxtaposition in $\mathcal{W}_\pi$ becomes multiplication in $\mathrm{St}(n, R)$; the empty word is mapped to the identity.

3.3. Definition. Let $A = A_0A_1A_2$ and $A' = A_0A_1'A_2$ be juxtapositions of three words. Then we say that A' is obtained from A by replacing the subword A_1 by A_1'. Only the following replacements and their compositions will be allowed (cf. (1.6)). (Recall that $\mathbf{0} = (0, 0, \ldots, 0)$, $\mathbf{1} = (1, 1, \ldots, 1)$ and that addition and multiplication of sequences is defined coordinatewise. Furthermore ( ) denotes the empty word.)

(1) $X_i(\mathbf{0}) \to (\ )$ for all i.
(2) $H(\mathbf{1}) \to (\ )$.
(3) $D \to (\ )$ if $D = 1$ in $D(R, I)$.
(4) $X_i(\boldsymbol{a})X_i(\boldsymbol{b}) \to X_i(\boldsymbol{a} + \boldsymbol{b})$ for all i.
(5) $H(\mathbf{u})H(\mathbf{v}) \to H(\mathbf{uv})\{u_1, v_1\}\{v_1, u_2\}\{u_2, v_2\} \cdot \cdots \cdot \{u_{n-1}, v_{n-1}\}$.
(6) $D_1D_2 \to (D_1D_2)$ where the right-hand side denotes the product in $D(R, I)$.
(7) $X_i(\mathbf{a})X_j(\mathbf{b}) \to X_i(\mathbf{c})X_j(\mathbf{d})H(\mathbf{u})\langle(-1)^{i+j+1}a_j, (-1)^{i+j+1}b_i\rangle$ whenever $i < j$, where

$$\begin{aligned}
c_k &= (b_k - a_kb_i)(1 + a_jb_i)^{-1} && \text{for } k \neq j,\\
d_k &= a_k + a_jb_k && \text{for } k \neq i, j,\\
d_j &= a_j(1 + a_jb_i),\\
u_k &= 1 + a_jb_i && \text{for } k = i, i+1, \ldots, j-1,\\
u_k &= 1 && \text{else.}
\end{aligned}$$

(8) $H(\mathbf{u})X_i(\mathbf{a}) \to X_i(\mathbf{b})H(\mathbf{u})$ for all i, where $b_k = a_ku_{k-1}u_k^{-1}u_{i-1}^{-1}u_i$ for $k = 1, \ldots, n$ (take $u_0 = u_n = 1$).
(9) $DL \to LD$ where $D \in D(R, I)$ and L is any $X_i(\mathbf{a})$ or $H(\mathbf{u})$.

The nine types of transformations listed above will be called elementary replacements. The type of an elementary replacement is its number in this list.

If B can be obtained from A by a replacement (not necessarily an elementary one and even $B = A$ is permitted) we will write $A \geqslant B$. It follows from (1.6) that $\beta_\pi(A) = \beta_\pi(B)$ if $A \geqslant B$.

3.4. Proposition. *$\geqslant$ is a partial order relation on $\mathcal{W}_\pi$. For all $A \in \mathcal{W}_\pi$ the set $\{B \in \mathcal{W}_\pi \mid A \geqslant B\}$ is finite.*

Proof. Consider for a word A the quintuple $r(A) = (r_1, r_2, r_3, r_4, r_5)$ defined by

r_1 = number of times one has $X_i(\mathbf{a})$ at the kth position in A (counted from left to right), $X_j(\mathbf{b})$ at the lth position while $i < j$ and $k < l$.

r_2 = number of times one has $X_i(\mathbf{a})$ at the kth position, $H(\mathbf{u})$ at the lth position while $k > l$.

r_3 = number of $H(\mathbf{u})$'s occurring in A.

r_4 = number of times one has D ($\in D(R, I)$) at the kth position and $X_i(\mathbf{a})$ or $H(\mathbf{u})$ at the lth position while $k < l$.

r_5 = the length of the word A.

The quintuples are ordered lexicographically. This definition of $r(A)$ was devised precisely so that $r(B) < r(A)$ if B can be obtained from A by an elementary replacement. From this the proposition easily follows. □

3.5. Definition. $\mathcal{M}_\pi$ is the set of minimal elements of $\mathcal{W}_\pi$ for the ordering $\geqslant$.

The elements of $\mathcal{M}_\pi$ are exactly the words of the form

$$X_n(\mathbf{a}^{(n)})X_{n-1}(\mathbf{a}^{(n-1)})\cdots X_1(\mathbf{a}^{(1)})H(\mathbf{u})D$$

where $X_k(\mathbf{a}^{(k)})$ (resp. $H(\mathbf{u})$ resp. D) only occurs if $\mathbf{a}^{(k)} \neq \mathbf{0}$ (resp. $\mathbf{u} \neq \mathbf{1}$ resp. $D \neq 1$). From this we see that β_π maps $\mathcal{M}_\pi$, and hence also $\mathcal{W}_\pi$, to $\tilde{\Omega}_n$ (see (1.5)).

3.6. Proposition. *For each $A \in \mathcal{W}_\pi$ there exists exactly one $B \in \mathcal{M}_\pi$ such that $A \geqslant B$.*

Proof. We show that the replacement system $\mathcal{W}_\pi$ has the so called Church–Rosser property:

If A, B and C are words in $\mathcal{W}_\pi$ such that $A \geqslant B$ and $A \geqslant C$, then there exists a word E such that $B \geqslant E$ and $C \geqslant E$.

This suffices to get the proposition: Let A be any word. There exists a word B in $\mathcal{M}_\pi$ such that $A \geqslant B$. If C is another such word, then according to the Church–Rosser property there will be a word E such that $B \geqslant E$ and $C \geqslant E$. By the minimality of B and C we find $B = E = C$.

Now we prove the Church–Rosser property for $\mathcal{W}_\pi$. Let $A, B, C \in \mathcal{W}_\pi$ be such that $A \geqslant B$ and $A \geqslant C$. We look for E such that $B \geqslant E$ and $C \geqslant E$. Since the set $\{F \in \mathcal{W}_\pi \mid A \geqslant F\}$ is finite, by an easy induction argument one can reduce the problem to the case where B and C are obtained from A by an elementary replacement. It is obvious that E exists if two disjoint subwords are replaced. E is also easily found if one of the two replacements is of type (1), (2), (3), (6) or (9). Since the replacements are now performed within a three letter subword of A, we may assume that A itself is a three letter word. Eight situations require further study. One can construct minimal words B_0 and C_0 such that $B \geqslant B_0$ and $C \geqslant C_0$. We indicate these constructions by a diagram. For instance

$$B - 1 - 3 - 2 - B_0$$

will indicate that one erases successively a letter $X_i(\mathbf{0})$, 1, $H(\mathbf{1})$ to get B_0; one might wonder which $X_i(\mathbf{0})$ should be erased if there are more than one, but that will not cause serious problems. Repeated use of 6 or 9 is indicated by just a single number 6 or 9. Since $\beta_\pi B_0 = \beta_\pi A = \beta_\pi C_0$, we see from (1.4) and (1.5) that the only difference between B_0 and C_0 can be in the letters from $D(R, I)$. So proving that B_0 and C_0 are equal comes down to proving some identity in $D(R, I)$.

We now list the eight cases.

I. $A = X_i(\mathbf{a})X_i(\mathbf{b})X_i(\mathbf{c})$.

$$A \begin{array}{l} \nearrow 4 - B - 4 - B_0 \\ \searrow 4 - C - 4 - C_0 \end{array}$$

There is nothing to be proved in $D(R, I)$.

II. $A = H(\mathbf{u})H(\mathbf{v})H(\mathbf{w})$.

$$A \begin{array}{l} \nearrow 5 - B - 9 - 5 - 6 - B_0 \\ \searrow 5 - C - 5 - 6 - C_0 \end{array}$$

To prove in $D(R, I)$:

$$\{u_1, v_1\}\{v_1, u_2\}\cdots\{u_{n-1}, v_{n-1}\}\{u_1v_1, w_1\}\cdots\{u_{n-1}v_{n-1}, w_{n-1}\} =$$
$$= \{u_1, v_1w_1\}\cdots\{u_{n-1}, v_{n-1}w_{n-1}\}\{v_1, w_1\}\cdots\{v_{n-1}, w_{n-1}\}.$$

This is an easy exercise.

III. $A = H(\mathbf{u})H(\mathbf{v})X_i(\mathbf{a})$.

$$A \begin{array}{l} \nearrow 5 - B - 9 - 8 - B_0 \\ \searrow 8 - C - 8 - 5 - C_0 \end{array}$$

The identity in $D(R, I)$ is trivial since it has equal left-hand and right-hand sides.

IV. $A = H(\mathbf{u})X_i(\mathbf{a})X_i(\mathbf{b})$.

$$A \begin{array}{l} \nearrow 8 - B - 8 - 4 - B_0 \\ \searrow 4 - C - 8 - C_0 \end{array}$$

There is nothing to be proved in $D(R, I)$.

V. $A = X_i(\mathbf{a})X_i(\mathbf{b})X_j(\mathbf{c})$ with $i < j$.

$$A \begin{array}{l} \nearrow 4 - B - 7 - B_0 \\ \searrow 7 - C - 7 - 9 - 8 - 4 - 5 - 6 - C_0 \end{array}$$

To prove in $D(R, I)$: (we write (ij) for $(-1)^{i+j+1}$)

$$\langle (ij)(a_j + b_j), (ij)c_i \rangle$$
$$= \langle (ij)b_j, (ij)c_i \rangle \left\langle (ij)a_j, (ij)\frac{c_i}{1 + b_jc_i} \right\rangle \left\{ \frac{1 + a_jc_i + b_jc_i}{1 + b_jc_i}, 1 + b_jc_i \right\}.$$

This follows immediately from (2.8), (D3) and (D2).

VI. $A = X_i(\mathbf{a})X_j(\mathbf{b})X_j(\mathbf{c})$ with $i<j$

$$A \begin{cases} 7 - B - 9 - 8 - 7 - 4 - 9 - 5 - 6 - B_0 \\ 4 - C - 7 - C_0 \end{cases}$$

To prove in $D(R, I)$:

$$\langle (ij)a_j, (ij)b_i \rangle \left\langle (ij)a_j(1+a_jb_i), (ij)\frac{c_i}{(1+a_jb_i)^2} \right\rangle \left\{ \frac{1+a_jb_i+a_jc_i}{1+a_jb_i}, 1+a_jb_i \right\}$$
$$= \langle (ij)a_j, (ij)(b_i+c_i) \rangle$$

which is easily done by (2.8), (D3) and (D2).

VII. $A = H(\mathbf{u})X_i(\mathbf{a})X_j(\mathbf{b})$ with $i<j$.

$$A \begin{cases} 8 - B - 8 - 7 - 9 - 5 - 6 - B_0 \\ 7 - C - 8 - 8 - 5 - 6 - C_0 \end{cases}$$

To prove in $D(R, I)$:

$$\langle (ij)a_ju_{i-1}^{-1}u_iu_{j-1}u_j^{-1}, (ij)b_iu_{j-1}^{-1}u_ju_{i-1}u_i^{-1} \rangle \cdot \{1+a_jb_i, u_{i-1}^{-1}u_{j-1}\} =$$
$$= \langle (ij)a_j, (ij)b_i \rangle \{1+a_jb_i, u_i^{-1}u_j\}.$$

This follows immediately from (D3) and (2.8).

VIII. $A = X_i(\mathbf{a})X_j(\mathbf{b})X_k(\mathbf{c})$ with $i<j<k$.

$$A \begin{cases} 7 - B - 9 - 8 - 7 - 9 - 5 - 7 - 9 - 5 - 8 - 6 - B_0 \\ 7 - C - 7 - 9 - 8 - 5 - 7 - 9 - 5 - 6 - C_0 \end{cases}$$

To prove in $D(R, I)$, writing $P = 1 + a_kc_i + a_jb_i + a_jb_kc_i$ and $Q = 1 + a_kc_i + b_kc_j - a_kb_ic_j$, that

$$\langle (ij)a_j, (ij)b_i \rangle \left\langle (ik)(a_k + a_jb_k), (ik)\frac{c_i}{1+a_jb_i} \right\rangle \{P, (1+a_jb_i)QP^{-1}\}$$
$$\cdot \left\langle (jk)\frac{b_k - a_kb_i}{1+a_jb_i}, (jk)\frac{(c_j - a_jc_i + a_jb_ic_j)(1+a_jb_i)}{P} \right\rangle \left\{ \frac{(1+a_jb_i)Q}{P}, \frac{P}{1+a_jb_i} \right\}$$

is equal to

$$\langle (jk)b_k, (jk)c_j \rangle \left\langle (ik)a_k, (ik)\frac{c_i - b_ic_j}{1+b_kc_j} \right\rangle \{Q(1+b_kc_j)^{-1}, 1+b_kc_j\}$$
$$\cdot \left\langle (ij)\frac{a_j + a_kc_j + a_jb_kc_j}{1+b_kc_j}, (ij)\frac{(b_i + b_kc_i)(1+b_kc_j)}{Q} \right\rangle \left\{ \frac{(1+b_kc_j)P}{Q}, \frac{Q}{1+b_kc_j} \right\}.$$

First we make the following substitutions in order to get rid of the signs (ij), (jk), (ik)

$$x_2 = (ij)a_j, \qquad x_3 = (ik)a_k,$$
$$y_1 = (ij)b_i, \qquad y_3 = (jk)b_k,$$
$$z_1 = (ik)c_i, \qquad z_2 = (jk)c_j,$$
$$R = 1 + x_3z_1 + x_2y_1 - x_2y_3z_1 \quad \text{and}$$
$$S = 1 + x_3z_1 + y_3z_2 + x_3y_1z_2.$$

Now by (D3) and (2.8) one has:

$$\left\langle \frac{y_3 + x_3y_1}{1 + x_2y_1}, \frac{(z_2 + x_2 z_1 + x_2y_1z_2)(1 + x_2y_1)}{R} \right\rangle \{(1 + x_2y_1)SR^{-1}, R(1 + x_2y_1)^{-1}\}$$
$$= \langle (y_3 + x_3y_1)R^{-1}, z_2 + x_2z_1 + x_2y_1z_2 \rangle.$$

Moreover

$$\{R, (1 + x_2y_1)SR^{-1}\} = \{R, -S\}\{1 + x_2y_1, -1\}\{R(1 + x_2y_1)^{-1}, (1 + x_2y_1)\}$$

while

$$\langle x_2, y_1 \rangle \{1 + x_2y_1, -1\} = \langle -x_2, -y_1 \rangle$$

and

$$\left\langle x_3 - x_2y_3, \frac{z_1}{1 + x_2y_1} \right\rangle \{R(1 + x_2y_1)^{-1}, 1 + x_2y_1\} = \left\langle \frac{x_3 - x_2y_3}{1 + x_2y_1}, z_1 \right\rangle.$$

Combining these four identities we see that the left-hand side of the desired identity is equal to

$$\langle -x_2, -y_1 \rangle \left\langle \frac{x_3 - x_2y_3}{1 + x_2y_1}, z_1 \right\rangle \langle (y_3 + x_3y_1)R^{-1}, z_2 + x_2z_1 + x_2y_1z_2 \rangle \{R, -S\}.$$

A similar calculation shows that the right-hand side is

$$\langle y_3, z_2 \rangle \left\langle x_3, \frac{z_1 + y_1z_2}{1 + y_3z_2} \right\rangle \langle -(x_2 - x_3z_2 + x_2y_3z_2), -(y_1 - y_3z_1)S^{-1} \rangle \{R, -S\}$$

The factors $\{R, -S\}$ cancel out. The rest of the left-hand side can be rewritten as follows

$$\langle -x_2, -y_1 \rangle \left\langle \frac{x_3 - x_2y_3}{1 + x_2y_1}, z_1 \right\rangle \langle (y_3 + x_3y_1)R^{-1}, z_2 + x_2z_1 + x_2y_1z_2 \rangle$$
$$= \langle -x_2, -y_1 \rangle \left\langle \frac{x_3 - x_2y_3}{1 + x_2y_1}, z_1 \right\rangle \langle (y_3 + x_3y_1)R^{-1}, x_2z_1 \rangle \left\langle (y_3 + x_3y_1)R^{-1}, \frac{z_2R}{1 + x_3z_1} \right\rangle$$
$$= \langle -x_2, -y_1 \rangle \left\langle \frac{x_3 - x_2y_3}{1 + x_2y_1}, z_1 \right\rangle \langle x_2(y_3 + x_3y_1)R^{-1}, z_1 \rangle \cdot$$
$$\cdot \langle z_1(y_3 + x_3y_1)R^{-1}, x_2 \rangle \left\langle (y_3 + x_3y_1)R^{-1}, \frac{z_2R}{1 + x_3z_1} \right\rangle$$

$$= \langle y_1, x_2\rangle^{-1}\langle x_3, z_1\rangle\langle z_1(y_3+x_3y_1)R^{-1}, x_2\rangle\left\langle (y_3+x_3y_1)R^{-1}, \frac{z_2R}{1+x_3z_1}\right\rangle$$

$$= \langle x_3, z_1\rangle\langle y_1, x_2\rangle^{-1}\left\langle \frac{-z_1(y_3+x_3y_1)}{(1+x_3z_1)(1+x_2y_1)}, x_2\right\rangle^{-1}\left\langle (y_3+x_3y_1)R^{-1}, \frac{z_2R}{1+x_3z_1}\right\rangle$$

$$= \langle x_3, z_1\rangle\left\langle \frac{y_1-y_3z_1}{1+x_3z_1}, x_2\right\rangle^{-1}\left\langle \frac{y_3+x_3y_1}{1+x_3z_1}, z_2\right\rangle\{S(1+x_3z_1)^{-1}, R(1+x_3z_1)^{-1}\}.$$

It is readily checked that all terms in the above computation are defined.

Now interchanging x_2 and z_2, y_1 and y_3, z_1 and $-x_3$ changes the first line of this computation to the inverse of the right-hand side of the desired identity, while the last line is changed to its own inverse. Now we are through. This is the end of the proof of Proposition (3.6) □

3.7. We are now in the position to make $\mathcal{M}_\pi$ into a group:

For $A, B \in \mathcal{M}_\pi$ define $A * B$ to be the unique element of $\mathcal{M}_\pi$ determined by the juxtaposition AB.

The unit element is the empty word ( ).

The inverse of $X_n(\mathbf{a}^{(n)})X_{n-1}(\mathbf{a}^{(n-1)})\cdots X_1(\mathbf{a}^{(1)})H(\mathbf{u})D$ is the minimal word that corresponds to

$$D^{-1}H(\mathbf{u}^{-1})X_1(-\mathbf{a}^{(1)})\cdots$$
$$\cdots X_n(-\mathbf{a}^{(n)})\{u_1u_2\cdots u_{n-1}, -1\}\{u_1, u_2\}\{u_2, u_3\}\cdots\{u_{n-2}, u_{n-1}\}.$$

It is clear that $D(R, I)$ is a subgroup of $\mathcal{M}_\pi$ and that $\beta_\pi : \mathcal{M}_\pi \to \mathrm{St}(n, R)$ is a homomorphism.

Let π' be another order relation on $\{1, 2, \ldots, n\}$ such that $\pi' \subseteq \pi$. Then $\mathcal{W}_{\pi'}$ is a subset of $\mathcal{W}_\pi$. The transformation rules on $\mathcal{W}_{\pi'}$ are restrictions of the transformation rules on $\mathcal{W}_\pi$. Moreover, if $A \in \mathcal{W}_{\pi'}$, $B \in \mathcal{W}_\pi$ and $A \geq B$ then $B \in \mathcal{W}_{\pi'}$.

All this shows that $\mathcal{M}_{\pi'}$ is a subgroup of $\mathcal{M}_\pi$. In particular every $\mathcal{M}_\pi$ contains the group $\mathcal{M}_\Delta$, where Δ is the order relation corresponding to the diagonal in $\{1, \ldots, n\} \times \{1, \ldots, n\}$.

3.8. Notation. We write $X_{ij}(a)$ for $X_i(\mathbf{a})$ with $\mathbf{a} = (0, \ldots, 0, a, 0, \ldots, 0)$, where a is on the jth spot. It is an element of $\mathcal{M}_\pi$ if $a \in I$ or if $(i, j) \in \pi$.

3.9. Lemma. *The following identities hold in every $\mathcal{M}_\pi$ that contains each term of the expression*

(i) $X_{ij}(a) * X_{ij}(b) = X_{ij}(a+b)$.

(ii) $[X_{ij}(a), X_{kl}(b)] = \begin{cases} (\ ) & \text{if } k \neq j \text{ and } l \neq i, \\ X_{il}(ab) & \text{if } k = j \text{ and } l \neq i. \end{cases}$

Proof. (i) Use an elementary replacement of type 4.

(ii) Use the following sequence of elementary replacements for the word

$X_{ij}(a) * X_{kl}(b) * X_{ij}(-a) * X_{kl}(-b)$:
If $k \neq j$ and $l \neq i$: 7 — 4 — 1 — 4 — 1.
If $k = j$, $l \neq i$ and $i < j$: 7 — 4 — 7 — 4 — 1.
If $k = j$, $l \neq i$ and $i > j$: 7 — 4 — 4 — 1. □

3.10. Lemma. *$\mathcal{M}_\Delta$ is a normal subgroup of every $\mathcal{M}_\pi$.*

Proof. It is easily seen that $\mathcal{M}_\pi$ is generated by $\mathcal{M}_\Delta$ and the elements $X_{ij}(a)$ with $(i,j) \in \pi$ and $a \in R$. Moreover $\mathcal{M}_\Delta$ itself is generated by the elements D, $H(\mathbf{u})$ (with the usual meaning) and $X_{ij}(a)$ with $a \in I$ and (i,j) arbitrary. Using (3.3) and (3.9) it is easily checked that $X_{ij}(a) * A * X_{ij}(a)^{-1} \in \mathcal{M}_\Delta$ for every generator A of $\mathcal{M}_\Delta$ and every $a \in R$ and $(i,j) \in \pi$. □

The proof of the first part of Theorem 3.1 comes to an end. The homomorphism β_Δ maps $\mathcal{M}_\Delta$ to a subgroup of $\mathrm{St}(n,R)$ which contains the elements $x_{ij}(a)$ with $a \in I$. Moreover, if $A \in \mathcal{M}_\Delta$ and $a \in R$, one can lift the expression $x_{ij}(a)\beta_\Delta(A)x_{ij}(a)^{-1}$ to $X_{ij}(a) * A * X_{ij}(-a)$ in some $\mathcal{M}_\pi$. Lemma 3.10 shows that the lifted expression is in $\mathcal{M}_\Delta$. So $x_{ij}(a)\beta_\Delta(A)x_{ij}(a)^{-1}$ is in the image of $\mathcal{M}_\Delta$. So the image of $\mathcal{M}_\Delta$ is a normal subgroup of $\mathrm{St}(n,R)$ which contains all $x_{ij}(a)$ with $a \in I$. Moreover $\beta_\Delta\mathcal{M}_\Delta \subset \ker(\mathrm{St}(n,R) \to \mathrm{St}(n,R/I))$. It is easily seen that there is a homomorphism $\mathrm{St}(n,R/I) \to \mathrm{St}(n,R)/\beta_\Delta\mathcal{M}_\Delta$ given by $x_{ij}(\bar{a}) \to x_{ij}(a) \bmod \beta_\Delta\mathcal{M}_\Delta$; here $\bar{a} \in R/I$ and a is representative for $\bar{a}$. This proves that $\beta_\Delta\mathcal{M}_\Delta = \mathrm{St}(n,R,I)$. Using the normal forms of (3.5) and (1.5) one sees that β_Δ maps the subgroup $D(R,I)$ onto the subgroup $K_2(n,R,I)$ of $\mathrm{St}(n,R,I)$. This finishes the proof of (3.1)(i).

From now on (R,I) is supposed to be a split radical pair. Write $S = R/I$. We exhibit an action of $\mathrm{St}(n,S)$ on $\mathcal{M}_\Delta$. For $x \in \mathrm{St}(n,S)$ and $A \in \mathcal{M}_\Delta$ the result of the action of x on A is denoted by ${}^xA$. For $x_{ij}(s) \in \mathrm{St}(n,S)$ and $A \in \mathcal{M}_\Delta$ define

$$ {}^{x_{ij}(s)}A = X_{ij}(s) * A * X_{ij}(-s), $$

computed in some $\mathcal{M}_\pi$ with $(i,j) \in \pi$. The right-hand side is in $\mathcal{M}_\Delta$ because of (3.10). Lemma 3.9 shows that this defines an action of $\mathrm{St}(n,S)$ on $\mathcal{M}_\Delta$. We make $\mathrm{St}(n,S)$ act on $\mathrm{St}(n,R,I)$ by conjugation. Then the map β_Δ commutes with the respective actions. Form the semi-direct product $\mathrm{St}(n,S) \cdot \mathcal{M}_\Delta$ according to this action. The elements of this group are pairs (x,A) with $x \in \mathrm{St}(n,S)$ and $A \in \mathcal{M}_\Delta$. The multiplication is defined by $(x,A)\cdot(y,B) = (xy, {}^{y^{-1}}A * B)$.

There is a homomorphism $\chi : \mathrm{St}(n,S) \cdot \mathcal{M}_\Delta \to \mathrm{St}(n,R)$ given by $\chi(x,A) = x \cdot \beta_\Delta(A)$. We construct an inverse to χ.

Each element of R can be written uniquely as $s + a$ with $s \in S$ and $a \in I$. Put $\psi(s+a) = (x_{ij}(s), X_{ij}(a))$.

It is clear that ψ will be inverse to χ, once it has been extended to all of $\mathrm{St}(n,R)$. The map ψ can be extended to all of $\mathrm{St}(n,R)$, if it preserves the relations (R1) and (R2) i.e. for $s,t \in S$ and $a,b \in I$ it should be that

$$(x_{ij}(s), X_{ij}(a))(x_{ij}(t), X_{ij}(b)) = (x_{ij}(s+t), X_{ij}(a+b))$$

and

$$[(x_{ij}(s), X_{ij}(a)), (x_{kl}(t), X_{kl}(b))] = \begin{cases} (1, (\ \ )) & \text{if } k \neq j \text{ and } l \neq i \\ (x_{il}(st), X_{il}(sb + ta + ab)) & \text{if } k = j \text{ and } l \neq i. \end{cases}$$

This can be checked by a straightforward computation. The restriction of ψ to $\mathrm{St}(n, R, I)$ is an inverse homomorphism to β_Δ. This shows that $\beta_\Delta : \mathcal{M}_\Delta \to \mathrm{St}(n, R, I)$ is an isomorphism. Using (1.5) and (3.5) one sees that the restriction of β_Δ to $D(R, I)$, that is δ, is an isomorphism $D(R, I) \xrightarrow{\sim} K_2(n, R, I)$. This concludes the proof of Theorem 3.1. □

3.11. Theorem 3.1(ii) makes it possible to give a presentation for Milnors relative group $K_2(I)$ for any radical pair (R, I). Let us first recall the definition of $K_2(I)$ (see [7, §6]). Consider a radical pair (R, I). Let T be the fibred product of R with itself over R/I i.e. $T = \{(x, x') \in R \times R \mid x \equiv x' \bmod I\}$ with multiplication and addition coordinatewise. Projection on the first factor $T \to R$ induces a morphism $K_2(T) \to K_2(R)$. By definition, $K_2(I)$ is the kernel of this morphism. Clearly, the kernel J of the projection $p_1 : T \to R$ is $\{(0, x) \mid x \in I\}$. This ideal is contained in the Jacobson radical of T. Furthermore, the projection p_1 is split by the diagonal map $x \to (x, x)$. So (T, J) is a split radical pair. What here is called $K_2(I)$ is what was called before $K_2(T, J)$. Hence we have a presentation for $K_2(I)$. Explicitly: Milnors $K_2(I)$ is isomorphic to the abelian group with generators $\langle (x, x'), (y, y') \rangle$ where $x \equiv x'$, $y \equiv y' \bmod I$ and x or $y = 0$, subject to the relations:

(D1) $\langle (x, x'), (y, y') \rangle = \langle -(y, y'), -(x, x') \rangle^{-1}$

(D2) $\langle (x, x'), (y, y') \rangle \langle (x, x'), (z, z') \rangle = \langle (x, x'), (y + z + xyz, y' + z' + x'y'z') \rangle$

(D3) $\langle (x, x'), (yz, y'z') \rangle = \langle (xz, x'z'), (y, y') \rangle \langle (xy, x'y'), (z, z') \rangle$.

3.12. Example. From (3.1) we will derive a result of Bloch [1, theorem 0.1]. Let R be a ring, S an augmented R-algebra with augmentation ideal J. Assume $J^N = 0$ for some $N \geq 1$. Moreover, assume that every positive integer $\leq N$ is invertible in R.

Write Ω^1_R and Ω^1_S for the groups of (absolute) Kähler differentials and $\Omega^1_{S,J}$ for the kernel of $\Omega^1_S \to \Omega^1_R$. Then there is a canonical isomorphism

$$K_2(S, J) \xrightarrow{\sim} \Omega^1_{S,J}/dJ.$$

This is proved via the presentation for $K_2(S, J)$ and $\Omega^1_{S,J}/dJ$. The group $\Omega^1_{S,J}/dJ$ is given by generators adb with $a, b \in S$ and a or b in J which are subject to the relations:

(Ω0) The group is commutative.

(Ω1) $1 \cdot da = 0$ for all $a \in J$.

(Ω2) $ad(b + c) = adb + adc$.

(Ω3) $ad(bc) = abdc + acdb$.

According to (3.1) $K_2(S,J)$ has a presentation with generators $\langle a,b\rangle$, for $a,b\in S$ and a or $b\in J$, and relations (D0), (D1), (D2) and (D3). We will use an additive notation.

First a special case:

$$K_2\left(\mathbb{Z}\left[\frac{1}{N!},T\right]\Big/(T^N),(T)\right)=0.$$

This will follow from the fact that for $m=1,\dots,N-1$ the canonical map

$$\phi_m: K_2(\mathbb{Z}[1/N!,T]/(T^{m+1}))\to K_2(\mathbb{Z}[1/N!,T]/(T^m))$$

is injective. According to (3.1) (i) the kernel of ϕ_m is generated by elements of the form $\langle fT^m,g\rangle$ with $f,g\in\mathbb{Z}[1/N!,T]/(T^{m+1})$. Since $T^{m+1}=0$ one can assume that $f\in\mathbb{Z}[1/N!]$. Write $g=g_0+g_1T$ with $g_0\in\mathbb{Z}[1/N!]$. Then:

$$\begin{aligned}\langle fT^m,g\rangle&=\langle fT^m\ g_0\rangle+\langle fT^m,g_1T\rangle\quad((\text{D2})\text{ and }T^{m+1}=0)\\&=\langle fT^m,g_0\rangle+\langle fg_1T^m,T\rangle\quad((\text{D3})\text{ and }T^{m+1}=0).\end{aligned}$$

Hence, $\ker\phi_m$ is generated by elements of the form $\langle fT^m,g\rangle$ and $\langle fT^m,T\rangle$ with $f,g\in\mathbb{Z}[1/N!]$.

Claim. All these elements are trivial.

As for $\langle fT^m,T\rangle$:

$$\begin{aligned}\langle fT^m,T\rangle&=(m+1)\langle fT^m,T/(m+1)\rangle&&((\text{D2})\text{ and }T^{m+1}=0)\\&=\langle f(m+1)^m,(T/(m+1))^{m+1}\rangle&&(\text{D3})\\&=0.\end{aligned}$$

As for $\langle fT^m,g\rangle$, it suffices to consider only $\langle fT,g\rangle\in K_2(\mathbb{Z}[1/N!,T]/(T^2))$, since one can use the substitution $T\mapsto T^m$.

The expression is linear in g, because of:

$$\begin{aligned}\langle fT,g\rangle+\langle fT,g'\rangle&=\langle fT,g+g'\rangle+\langle fT,fgg'T\rangle\\&=\langle fT,g+g'\rangle+\langle fT^2,fgg'\rangle+\langle f^2gg'T,T\rangle\\&=\langle fT,g+g'\rangle.\end{aligned}$$

Now, let n be a power of $N!$ such that $ng\in\mathbb{Z}$. Then:

$$\langle fT,g\rangle=n\langle fT/n,g\rangle=\langle fT/n,ng\rangle=ng\langle fT/n,1\rangle=0.$$

This concludes the treatment of the special case.

Our only reasons for treating this special case are the formulas

$$\langle 1,T\rangle=0\quad\text{and}\quad\langle e(T,1),T\rangle=0\quad\text{in }K_2(\mathbb{Z}[1/N!,T]/(T^N)),$$

where $e(T,1)$ is defined below.

$X^{-1}(\exp(XY)-1)$, which a priori is an element of $\mathbb{Q}[\![X,Y]\!][X^{-1}]$, lies in $\mathbb{Q}[\![X,Y]\!]$. Its image under the canonical projection onto $\mathbb{Q}[\![X,Y]\!]/((XY)^N)$ is an element of $\mathbb{Q}[1/N!,X,Y]/((XY)^N)$, which we denote by $e(X,Y)$.

Similarly, let $l(X, Y)$ be the image of $X^{-1}\log(XY+1)$ in $\mathbb{Z}[1/N!, X, Y]/((XY)^N)$. From this definition it is obvious that in $\mathbb{Z}[1/N!, X, Y, Z]/(I^N)$, where $I=(XY, XZ)$, the following identities hold.

$e(X, Y+Z) = e(X, Y) + e(X, Z) + Xe(X, Y)e(X, Z)$

$l(X, Y) + l(X, Z) = l(X, Y+Z+XYZ)$

$e(X, YZ) = Ye(XY, Z)$

$l(X, YZ) = Yl(XY, Z)$

$e(X, l(X, Y)) = Y$

$l(X, e(X, Y)) = Y.$

Now everything is ready for the definition of the isomorphisms for general S and J. Put

$\sigma\langle a, b\rangle = l(a, b)da, \qquad \tau(adb) = \langle e(a, b), a\rangle.$

By the special case above $\langle e(ab, 1), ab\rangle = 0$, whence $\langle e(a, b), a\rangle = -\langle e(b, a), b\rangle$. Similarly $\langle a, b\rangle = -\langle b, a\rangle$.

Using this and the identities above it is easily checked that τ respects the relations (Ω0)–(Ω3). Hence τ gives a homomorphism

$$\tau : \Omega^1_{S,J}/dJ \to K_2(S, J).$$

That σ respects (D1) follows from

$$l(a, b)da + l(-b, -a)d(-b) = l(ab, 1)d(ab) = d\Big(-\sum_{n=1}^{N}(-1)^n n^{-2}(ab)^n\Big).$$

That σ respects (D2) is obvious from the identities again. To see that (D3) is also preserved one uses the fact that $l(a, b)da = l(-b, -a)db$. So σ extends to a homomorphism $K_2(S, J) \to \Omega^1_{S,J}/dJ$.

We leave it to the reader to check that σ and τ are inverse to each other.

Remark. Leslie Roberts pointed out that it is not sufficient that $(N-1)!$ is a unit, as we stated in a previous version. His counter example: $R = \mathbb{F}_p$, $S = \mathbb{F}_p[T]/(T^p)$.

4. A reduction theorem

The main result of this section is a theorem which states that if $f : S \to R$ is a ring homomorphism, and $\delta : D(S) \to K_2(n, S)$ is an isomorphism, then under certain conditions $\delta : D(R) \to K_2(n, R)$ is an isomorphism too. In this section, it is always assumed $3 \leq n \leq \infty$. (Convention: $K_2(\infty, R) = K_2(R)$, etc.)

4.1. Proposition. (i) *If (R, I) is a radical pair such that $\delta : D(R) \to K_2(n, R)$ is an isomorphism, then $\delta : D(R/I) \to K_2(n, R/I)$ is an isomorphism too.*
(ii) *If the radical pair (R, I) is split and $\delta : D(R/I) \to K_2(n, R/I)$ is an isomorphism then $\delta : D(R) \to K_2(n, R)$ is an isomorphism.*

Proof. Since $I \subset \operatorname{rad}(R)$ we have $\mathrm{St}(n,R,I) \subset \tilde{\Omega}_n$ and $E(n,R,I) \subset \Omega_n$ (cf. (1.3)). Using (1.4) and (1.5) one immediately sees that $\mathrm{St}(n,R,I) \to E(n,R,I)$ is surjective. Applying the snake lemma to the diagram

$$\begin{tikzcd}
 & & K_2(n,R) \arrow[r] \arrow[d] & K_2(n,R/I) \arrow[d] \arrow[dddll, dashed, bend left] & \\
1 \arrow[r] & \mathrm{St}(n,R,I) \arrow[r] \arrow[d] & \mathrm{St}(n,R) \arrow[r] \arrow[d] & \mathrm{St}(n,R/I) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & E(n,R,I) \arrow[r] \arrow[d] & E(n,R) \arrow[r] & E(n,R/I) \arrow[r] & 1 \\
 & 1 & & &
\end{tikzcd}$$

we find that $K_2(n,R) \to K_2(n,R/I)$ is surjective. In the following diagram with exact rows the top row is exact by (2.11)(iii):

$$\begin{tikzcd}
 & D(R,I) \arrow[r] \arrow[d, "\delta"] & D(R) \arrow[r] \arrow[d, "\delta"] & D(R/I) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & K_2(n,R,I) \arrow[r] & K_2(n,R) \arrow[r] & K_2(n,R/I) \arrow[r] & 0.
\end{tikzcd}$$

The vertical arrow on the left is surjective by (3.1)(i). In case (ii) it is an isomorphism by (3.1)(ii). Now use the snake lemma. □

4.2. Theorem. *Let (R,I) be a radical pair. Let S be a ring, $f: S \to R$ a homomorphism such that*

(i) $R = f(S) + I$

(ii) $(S, f^{-1}(I))$ *is a radical pair.*

Suppose that $\delta: D(S) \to K_2(n,S)$ is an isomorphism. Then $\delta: D(R) \to K_2(n,R)$ is an isomorphism too.

Proof. Let the ring T be defined as follows. As a set $T = S \times I$. Addition is defined coordinatewise, and we set

$$(s,i)(s',i') = (ss', f(s)i' + f(s')i + ii')$$

for $s, s' \in S$, $i, i' \in I$.

Projection onto the first factor defines a surjective homomorphism $p: T \to S$. Let

$$J = \operatorname{Ker} p = \{(0,i) \mid i \in I\}$$

be its kernel. Since $I \subset \operatorname{Rad}(R)$ the pair (T,J) is a radical pair. As p is split by the map $s \mapsto (s,0)$ and $\delta: D(S) \to K_2(n,S)$ is an isomorphism, it follows from (4.1)(ii) that $\delta: D(T) \to K_2(n,T)$ is an isomorphism.

Now define $g: T \to R$ by $(s,i) \to f(s) + i$. By (i) g is surjective and

$$K = \operatorname{Ker} g = \{(j, -f(j)) \mid j \in f^{-1}(I)\}.$$

Condition (ii) shows that (T,K) is a radical pair. From (4.1)(i) we conclude that $\delta: D(R) \to K_2(n,R)$ is an isomorphism. □

4.3. Example. If S is a ring such that $\delta : D(S) \to K_2(n, S)$ is an isomorphism then also $\delta : D(S[\![X_1, \ldots, X_n]\!]) \to K_2(n, S[\![X_1, \ldots, X_n]\!])$ is an isomorphism for $3 \leq n \leq \infty$. This follows for instance from (4.1)(ii).

5. K_2 of discrete valuation rings and applications

In their paper [3] Dennis and Stein give a presentation of K_2 of a discrete valuation ring. Using the symbols $\langle\ ,\ \rangle$ we are able to give a much more simple presentation of this K_2. We also draw some conclusions from Theorem 4.2.

5.1. Theorem. (Dennis and Stein [3, theorem 2.3]). *Let A be a discrete valuation ring with maximal ideal P. Then for every integer $n \geq 3$, including $n = \infty$, $K_2(n, A)$ is isomorphic to the abelian group with generators $\{u, v\}$, $u, v \in A^*$, subject to the relations*

(S1) $\{u_1u_2, v\} = \{u_1, v\}\{u_2, v\}$

(S2) $\{u, v\} = \{v, u\}^{-1}$

(S3) $\{u, -u\} = 1$

(S4) $\{u, 1-u\} = 1$ *if* $1-u \in A^*$

(S5) $\{v, 1-pqv\} = \left\{-\dfrac{1-qv}{1-p}, \dfrac{1-pqv}{1-p}\right\}\left\{-\dfrac{1-pv}{1-q}, \dfrac{1-pqv}{1-q}\right\}$ *for* $p, q \in P$

(S6) $\left\{-\dfrac{1-qr}{1-p}, \dfrac{1-pqr}{1-p}\right\}\left\{-\dfrac{1-pr}{1-q}, \dfrac{1-pqr}{1-q}\right\}\left\{-\dfrac{1-pq}{1-r}, \dfrac{1-pqr}{1-r}\right\} = 1$

for $p, q, r \in P$

(S7) $\{u_1, 1+qu_1\}\left\{\dfrac{u_2}{1+qu_1}, \dfrac{1+q(u_1+u_2)}{1+qu_1}\right\} = \{v_1, 1+qv_1\}\left\{\dfrac{v_2}{1+qv_1}, \dfrac{1+q(v_1+v_2)}{1+qv_1}\right\}$

for all $q \in P$, $u_1, u_2, v_1, v_2 \in A^*$ *such that* $u_1 + u_2 = v_1 + v_2 = P$.

The isomorphism is given by sending the generator $\{u, v\}$ to the Steinberg symbol $\{u, v\}_* \in K_2(n, A)$. As a consequence

$$K_2(n, A) \xrightarrow{\sim} K_2(n+1, A) \xrightarrow{\sim} K_2(A)$$

for all $n \geq 3$. □

5.2. Theorem. *Let A be a discrete valuation ring. Then for $3 \leq n \leq \infty$ the map $\delta : D(A) \to K_2(n, A)$ is an isomorphism, i.e. $K_2(n, A)$ is isomorphic to the abelian group with generators $\langle a, b\rangle$, $a, b \in A$, $1 + ab \in A^*$ and relations*

(D1) $\langle a, b\rangle = \langle -b, -a\rangle^{-1}$

(D2) $\langle a, b\rangle\langle a, c\rangle = \langle a, b+c+abc\rangle$

(D3) $\langle a, bc\rangle = \langle ab, c\rangle\langle ac, b\rangle$.

Proof. Identify $K_2(n, A)$ with the group described in (5.1). We are going to construct an inverse ϕ of δ. Of course we define $\phi\{u, v\} = \{u, v\}$. To prove ϕ is "good", we must verify (S1)–(S7) in $D(A)$. Relations (S1)–(S4) are proved in (2.8). To prove relations (S5) and (S6) we must use

$$(*)\qquad \left\{-\frac{1-a}{1-r},\frac{1-ar}{1-r}\right\}=\langle -a,r\rangle \quad \text{if} \quad a\in A,\quad r\in P$$

in which P is again the maximal ideal of A. This relation is proved as follows:

$$\begin{aligned}\langle -a,r\rangle &= \langle -1+1-a,r\rangle \\ &= \langle -1,r\rangle\left\langle\frac{1-a}{1-r},r\right\rangle \\ &= \left\{-\frac{1-a}{1-r},\frac{1-ar}{1-r}\right\}.\end{aligned}$$

Note that everything is defined because $r\in P$. Using $(*)$ one sees relation (S5) reduces to

$$\langle -v,pq\rangle=\langle -qv,p\rangle\langle -pv,q\rangle$$

which is a special case of (D3) and (S5) reduces to

$$\langle -qr,p\rangle\langle -p,q\rangle\langle -pq,r\rangle=1$$

which is easily seen to be a special case of (D3) too. To prove (S7) write the left-hand side as

$$\begin{aligned}\{u_1,1+qu_1\}\left\{\frac{u_2}{1+qu_1},\frac{1+q(u_1+u_2)}{1+qu_1}\right\} &= \langle -u_1,-q\rangle\left\langle -\frac{u_2}{1+qu_1},-q\right\rangle \\ &= \langle -(u_1+u_2),-q\rangle.\end{aligned}$$

Similarly, the right-hand side is equal to $\langle -(v_1+v_2),-q\rangle$. As $u_1+u_2=v_1+v_2$, both sides are equal, and (S7) is proved in $D(A)$. As $K_2(n,A)$ is generated by Steinberg symbols, it follows from the definition of ϕ and (2.9) that $\delta\phi=\mathrm{id}_{K_2(n,A)}$. Now δ is surjective, so δ is an isomorphism if we prove that ϕ is surjective.

Let $\langle a,b\rangle$ be a generator of $D(A)$. Now one of the following cases holds:

(i) $b\in A^*$ then $\langle a,b\rangle=\{1+ab,b\}\in\operatorname{im}\phi$

(ii) $b\in P$ then $\langle a,b\rangle=\left\{-\frac{1+a}{1-b},\frac{1+ab}{1-b}\right\}\in\operatorname{im}\phi$.

This proves ϕ is surjective and the theorem is proved. □

5.3. Remark. One can also prove Theorem 5.2 without using (5.1). For that, one must substitute relations (S1)–(S7) in Dennis and Stein's proof by (D1)–(D3). We omit the details.

5.4. Proposition. *Let R be a local ring, with residue class field k. Let S be a discrete valuation ring, $f:S\to R$ a homomorphism, such that the composite map $S\to R\to k$ is surjective. Then $\delta:D(R)\to K_2(n,R)$ is an isomorphism for $3\leqslant n\leqslant\infty$.*

Proof. By (5.2), $\delta : D(S) \to K_2(n, S)$ is an isomorphism. It is easy to see that the radical pair (R, P), P the maximal ideal of R, satisfies the conditions of Theorem 4.2. □

5.5. Theorem. *Let R be a local ring, such that the residue class field k is isomorphic to $\mathbb{F}_p$, p prime. Then $\delta : D(R) \to K_2(n, R)$ is an isomorphism.*

Proof. We have a homomorphism $\mathbb{Z} \to R$. Because R is a local and $k \cong \mathbb{F}_p$ this extends to a homomorphism $\mathbb{Z}_{(p)} \to R$ and evidently $\mathbb{Z}_{(p)} \to R \to \mathbb{F}_p$ is surjective. Now use (5.4). □

5.6. Let R be a local ring, k its residue class field. We say that k has a multiplicative system of representatives if the projection $R^* \to k^*$ has a section which is a homomorphism.

We denote by ζ_d a primitive dth root of unity in $\mathbb{C}$. Its minimal polynomial over $\mathbb{Q}$ is denoted by $\Phi_d(X)$. It is an element of $\mathbb{Z}[X]$. Its image in any ring is also denoted by $\Phi_d(X)$.

Theorem. *Let R be a local ring with finite residue class field k of order q. If R has a multiplicative system of representatives, then $\delta : D(R) \to K_2(n, R)$ is an isomorphism for $3 \leq n \leq \infty$.*

Proof. The group k^* is cyclic. Let a be a generator, α its representative. Let $m = q - 1$. If $d \mid m$, $d \neq m$ then $a^d - 1 \neq 0$. As $a^d - 1 = \Pi_{f|d} \Phi_f(a)$, it follows that $\Phi_d(a) \neq 0$, and hence $\Phi_d(\alpha) \in R^*$. Now

$$0 = \alpha^m - 1 = \prod_{d|m} \Phi_d(\alpha).$$

We conclude that $\Phi_m(\alpha) = 0$. From this we see that the homomorphism $\mathbb{Z}[X] \to R$ which sends X to α factors over $\mathbb{Z}[X]/(\Phi_m(X)) \cong \mathbb{Z}[\zeta_m]$. Now

$$\mathbb{Z}[\zeta_m] \to R \to k$$

is surjective since ζ_m goes to α. Let $\mathcal{Y} = \mathrm{Ker}(\mathbb{Z}[\zeta_m] \to k)$. Since R is local, we have a homomorphism

$$\mathbb{Z}[\zeta_m]_{\mathcal{Y}} \to R.$$

By [9, theorem 7–5–4], $\mathbb{Z}[\zeta_m]$ is the ring of integers in $\mathbb{Q}(\zeta_m)$, hence it is a dedekind domain. As the height of $\mathcal{Y}$ is one, $\mathbb{Z}[\zeta_m]_{\mathcal{Y}}$ is a discrete valuation ring. Now use (5.4). □

5.7. Corollary. *If R is a complete local ring with finite residue class field then $\delta : D(R) \to K_2(n, R)$ is an isomorphism.*

Proof. R has a multiplicative system of representatives by [2, lemma 7]. □

Added in proof. Frans Keune recently generalized the result of Theorem (3.1) as follows: If (R, I) is a radical pair then there is an exact sequence

$$K_3(R) \to K_3(R/I) \to D(R, I) \to K_2(R) \to K_2(R/I) \to 0,$$

where however $D(R, I)$ is defined using relation (D3) only with a or b or $c \in I$ (also see Remark (2.3)).

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